Theorem: 8

Let $\varphi_1, \varphi_2 \dots \varphi_n$ be n solns of $L(y)=0$ on an interval I and let x_0 be any point in I. Then,

$$W(\varphi_1,\varphi_2 \dots \varphi_n)(x) = \exp\left[-\int_{x_0}^{x} a_1(t)\,dt\right] W(\varphi_1,\varphi_2 \dots \varphi_n)(x_0)$$

Proof:

We first prove this result for the simple case for $n=2$

Consider $L(y) = y'' + a_1(x)y' + a_2(x)y = 0$.

Since φ_1, φ_2 are soln of $L(y)=0$.

$$L(\varphi_1) = \varphi_1'' + a_1(x)\varphi_1' + a_2(x)\varphi_1 = 0 \rightarrow (1)$$

$$L(\varphi_2) = \varphi_2'' + a_1(x)\varphi_2' + a_2(x)\varphi_2 = 0 \rightarrow (2)$$

$(1) \times \varphi_2 \Rightarrow \varphi_1''\varphi_2 + a_1\varphi_1'\varphi_2 + \cancel{a_2\varphi_1\varphi_2} = 0$

$(2) \times \varphi_1 \Rightarrow \underset{(-)}{\varphi_2''\varphi_1} + \underset{(-)}{a_1\varphi_2'\varphi_1} + \underset{(-)}{\cancel{a_2\varphi_1\varphi_2}} = 0$

$$(\varphi_1''\varphi_2 - \varphi_2''\varphi_1) + (a_1\varphi_1'\varphi_2 - a_1\varphi_2'\varphi_1) = 0$$

$$\varphi_1''\varphi_2 - \varphi_2''\varphi_1 = -a_1(\varphi_1'\varphi_2 - \varphi_1\varphi_2') \rightarrow (3)$$

Now,

$$W = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix} = \varphi_1\varphi_2' - \varphi_2\varphi_1'$$

$$W = \varphi_1\varphi_2' - \varphi_1'\varphi_2$$

$$W' = \varphi_1'\varphi_2' + \varphi_1\varphi_2'' - \varphi_1''\varphi_2 - \varphi_1'\varphi_2'$$

$$W' = \varphi_1\varphi_2'' - \varphi_1''\varphi_2 \Rightarrow -(\varphi_1''\varphi_2 - \varphi_1\varphi_2'')$$

$\therefore (3) \Rightarrow$ $-W' = a_1 W$

$$W' = -a_1 W$$

$$\frac{W'}{W} = -a_1$$

$W(\varphi_1,\varphi_2)$ satisfying the 1st linear eqn.

$$\log \frac{W(x)}{W(x_0)} = -\int a_1(t)\,dt$$

$$W(x) = \exp\left[-\int_{x_0}^{x} a_1(t)\,dt\right] W(x_0)$$

(i) $W(\varphi_1,\varphi_2)(x) = \exp\left[-\int_{x_0}^{x} a_1(t)\,dt\right] W(\varphi_1,\varphi_2)(x_0)$

Proof at for general case n

let $w = w(\varphi_1, \varphi_2 \dots \varphi_n)$

$$\Rightarrow w(x) = \begin{vmatrix} \varphi_1 & \varphi_2 & \dots & \varphi_n \\ \varphi_1' & \varphi_2' & \dots & \varphi_n' \\ \vdots & & & \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} & \dots & \varphi_n^{(n-1)} \end{vmatrix}$$

from The above definition of w as a determinant it follows that is derivative w' is a sum of n derivative

$$w' = V_1 + V_2 + \dots + V_n$$

Where V_k difference from w only in its k^{th} row and the k^{th} row of V_k is obtained by differentiating The k^{th} row of w

Thus,

$$w' = \begin{vmatrix} \varphi_1 & \varphi_2 \dots & \varphi_n \\ \varphi_1' & \varphi_2' \dots & \varphi_n' \\ \vdots & & \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} \dots & \varphi_n^{(n-1)} \end{vmatrix} + \begin{vmatrix} \varphi_1 & \varphi_2 \dots & \varphi_n \\ \varphi_1' & \varphi_2' \dots & \varphi_n' \\ \vdots & & \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} \dots & \varphi_n^{(n-1)} \end{vmatrix} + \dots + \begin{vmatrix} \varphi_1 & \varphi_2 \dots & \varphi_n \\ \varphi_1' & \varphi_2' \dots & \varphi_n' \\ \vdots & & \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} \dots & \varphi_n^{(n-1)} \end{vmatrix}$$

The first $n-1$ determinants $V_1, V_2 \dots V_{n-1}$ are all zero. Since they each have two identical rows.

Since $\varphi_1, \varphi_2 \dots \varphi_n$ are Solns of $L(y)=0$

Wehave,

$$\varphi_i^{(n)} = -a_1\varphi_i^{(n-1)} - a_2\varphi_i^{(n-2)} \dots - a_n\varphi_i \qquad (i=1,2\dots n)$$

$$\therefore w' = \begin{vmatrix} \varphi_1 & \varphi_2 & \dots & \varphi_n \\ \varphi_1' & \varphi_2' & \dots & \varphi_n' \\ \vdots & & & \\ \varphi_1^{(n-2)} & \varphi_2^{(n-2)} & \dots & \varphi_n^{(n-2)} \\ -\sum_{j=0}^{n-1} a_{n-j}\varphi_1^{(j)} & \dots & \dots & -\sum_{j=0}^{n-1} a_{n-j}\varphi_n^{(j)} \end{vmatrix}$$

The value of This determinant is unchanged if we multiply any row by a x_0 and add to the last row

We multiply the first row by a_n ...

Then $(n-1)^{th}$ row by a_2 and add these to the last row

thus we obtain,

$$\omega' = \begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1' & \varphi_2' & \cdots & \varphi_n' \\ \vdots & & & \\ \varphi_1^{(n-2)} & \varphi_2^{(n-2)} & \cdots & \varphi_n^{(n-2)} \\ -a_1\varphi_1^{(n-1)} & & \cdots & -a_1\varphi_n^{(n-1)} \end{vmatrix} = -a_1\omega$$

$\therefore$ W satisfies the linear first order eqn, $y' + a_1(x)y = 0$.

$$\frac{\omega'}{\omega} = -a_1$$

$$\log \frac{\omega(x)}{\omega(x_0)} = -\int_{x_0}^{x} a_1(t)\,dt$$

$$\Rightarrow \omega(x) = \exp\left[-\int_{x_0}^{x} a_1(t)\,dt\right]\omega(x_0)$$

ie $\omega(\varphi_1, \varphi_2 \ldots \varphi_n)(x) = \exp\left[-\int_{x_0}^{x} a_1(t)\,dt\right]\omega(\varphi_1, \varphi_2 \ldots \varphi_n)(x_0)$

Hence the theorem.

Corollary:

If the coefficient a_k of L are constants

$$\omega(\varphi_1, \varphi_2 \ldots \varphi_n)(x) = e^{-a_1(x-x_0)}\,\omega(\varphi_1, \varphi_2 \ldots \varphi_n)(x_0)$$

Note:

A sequence of above thm, is that n solns, $\varphi_1, \varphi_2 \ldots \varphi_n$ of $L(y) = 0$ on an interval I are L.I iff $\omega(\varphi_1, \varphi_2 \ldots \varphi_n)(x_0) \neq 0$. for any particular x_0 in I.

2.a) One soln of $L(y) = y'' + \frac{1}{4x^2}y = 0$ for $x>0$ is $\varphi(x) = x^{1/2}$. S.T there is another soln ψ of the form $\psi = u\varphi$ where u is some function

Soln:

Given eqn $L(y) = y'' + \frac{1}{4x^2}y = 0 \rightarrow (1)$ for $x>0$. and $\varphi(x) = x^{1/2}$

Now,

$\psi = u\varphi \Rightarrow ux^{1/2}$

$\psi' = u \cdot \frac{1}{2}x^{-1/2} + x^{1/2}u'$

$\varphi' = \frac{1}{2} u x^{-1/2} + x^{1/2} u'$

$\Rightarrow \varphi'' = \frac{1}{2} u' x^{-1/2} + \frac{1}{2} u(-1/2) x^{-3/2} + x^{1/2} u'' + u'(1/2) x^{-1/2}$

Since φ is a soln,

$$\varphi'' + \frac{1}{4x^2}\varphi = 0$$

$$\Rightarrow x^{1/2} u'' + u' x^{-1/2} - \frac{1}{4} u x^{-3/2} + \frac{1}{4x^2} u x^{1/2} = 0$$

$$\Rightarrow 4x^2 x^{1/2} u'' + 4x^2 x^{-1/2} u' - x^{-3/2} x^2 u + u x^{1/2} = 0$$

$$\Rightarrow 4x^{5/2} u'' + 4x^{3/2} u' - x^{1/2} u + x^{1/2} u = 0$$

$$\Rightarrow 4x^{5/2} u'' + 4x^{3/2} u' = 0$$

$$x^{5/2} u'' + x^{3/2} u' = 0$$

$$x^{3/2}(x u'' + u') = 0$$

$$x u'' + u' = 0$$

$$x u'' = -u' \Rightarrow u'' = -u'/x$$

$$\frac{u''}{u'} = -\frac{1}{x}$$

Integrating we get,

$$\log u' = -\log x$$

$$\log u' = \log x^{-1}$$

$$u' = 1/x$$

Integrating again we get,

$$u = \log x$$

$$\varphi = u\varphi = x^{1/2} \log x$$

2.b Consider the eqn $L(y) = y'' + a_1(x) y' + a_2(x) y = 0$. Where a_1, a_2 are continuous on some interval I. S.T a_1, a_2 are uniquely determined by any basis φ_1, φ_2 for the soln of $L(y)=0$.

Soln:

Given eqn is $L(y) = y'' + a_1(x) y' + a_2(x) y = 0$

We have $L(\varphi_1) = 0$, $L(\varphi_2) = 0$ since φ_1, φ_2 are soln of $L(y)=0$

(i) $\varphi_1'' + a_1(x)\varphi_1' + a_2(x)\varphi_1 = 0 \rightarrow (1)$

$\varphi_2'' + a_1(x)\varphi_2' + a_2(x)\varphi_2 = 0 \rightarrow (2)$

a_1 a_2 Const

$(1) \Rightarrow \quad a_1(x)\varphi_1' + a_2(x)\varphi_1 = -\varphi_1''$ $\qquad \phi_1' \quad \phi_1 \quad -\phi_1''$

$(2) \Rightarrow \quad a_1(x)\varphi_2' + a_2(x)\varphi_2 = -\varphi_2''$ $\qquad \phi_2' \quad \phi_2 \quad -\phi_2''$

Since the coefficient determinant is just $W(\varphi_1, \varphi_2) \neq 0$, we get unique values for a_1 and a_2

$$a_1 = -\frac{\begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1'' & \varphi_2'' \end{vmatrix}}{W(\varphi_1, \varphi_2)} \quad \text{and} \quad a_2 = \frac{\begin{vmatrix} \varphi_1' & \varphi_2' \\ \varphi_1'' & \varphi_2'' \end{vmatrix}}{W(\varphi_1, \varphi_2)}$$

Section-5

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Reduction of the order of a homogeneous eqn

Suppose we have found by some means one soln φ_1 of the eqn

$$L(y) = y^{(n)} + a_1(x)y^{(n-1)} + \dots + a_n(y) = 0$$

It is then possible to reduce the ~~centre~~ order of the eqn to be solved by one. The idea is the same as in case of the variation of constants method.

We try to find soln φ of $L(y)=0$ of the form $\varphi = u\varphi_1$, where u is some function.

Theorem: 9

Let φ_1 be a soln of $L(y)=0$ on an interval I and suppose $\varphi_1(x) \neq 0$ on I. If $v_2 \dots v_n$ is any basis on I for the solns of the linear eqn

$$\varphi_1 v^{(n-1)} + \dots + (n\varphi_1^{(n-1)} + (n-1)\varphi_1^{(n-2)} + \dots + a_{n-1}\varphi_1)v = 0$$

of order $n-1$ and if $v_k = u_k'$ $(k = 2 \dots n)$ then $\varphi_1, u_2\varphi_1, \dots, u_n\varphi_n$ is a basis for the solns of $L(y)=0$ on I.

Proof:

Now $L(y) = y^{(n)} + a_1(x) y^{(n-1)} + \dots + a_n(x) y = 0$.

Also given φ_1 is one soln of $L(y)=0$ on I and $\varphi_1(x) \neq 0$ on I.

It is then possible to reduce the order of $L(y)=0$ of the form $\varphi = u\varphi_1$, where u is some function of x.

If $\varphi = u\varphi_1$ is a soln, we must have,

$$(u\varphi_1)^{(n)} + a_1 (u\varphi_1)^{(n-1)} + \dots + a_{n-1}(u\varphi_1)' + a_n(u\varphi_1) = 0$$

$$\text{ie) } \Big\{ [u^{(n)}\varphi_1 + \dots + u\varphi_1^{(n)}] + a_1[u^{(n-1)}\varphi_1 + \dots + u\varphi_1^{(n-1)}] + \dots + a_{n-1}[u'\varphi_1 + u\varphi_1'] + a_n u\varphi_1 \Big\} = 0$$

equation is $L(\varphi_1) = 0$. ∴ If $v = u'$, this equation

The coefficient of u in the above reduces to a linear eqn of order $(n-1)$ in v.

$$\text{ie) } \varphi_1 v^{(n-1)} + \dots + [n\varphi_1^{(n-1)} + (n-1)\varphi_1^{(n-2)} + \dots + a_{n-1}\varphi_1] v = 0 \quad \to (1)$$

The coefficient of $v^{(n-1)}$ is φ_1

Also $\varphi_1(x) \neq 0$ on an interval I.

⇒ This eqn has $(n-1)$ L.I soln $v_2 \dots v_n$ on I.

If x_0 is some point in I and

$$u_k(x) = \int_{x_0}^{x} v_k(t)\,dt, \quad k = 2, 3 \dots n$$

Then we have $u_k' = v_k$ and the function $\varphi_1, u_2\varphi_1, \dots u_n\varphi_1$ are soln of $L(y)=0$ $\longrightarrow (2)$

Moreover these function forms basis for the soln of $L(\varphi_1)=0$ on I.

for suppose we have constants $c_1, c_2 \dots c_n$ such that.

$$c_1\varphi_1 + c_2 u_2\varphi_1 + \dots + c_n u_n \varphi_1 = 0$$

Since $\varphi_1(x) \neq 0$ on I

We have

$$c_1 + c_2 u_2 + \dots + c_n u_n = 0 \quad \to (3)$$

Diff we get,

$c_2 u_2' + \cdots + c_n u_n' = 0$

$\Rightarrow c_2 v_2 + \cdots + c_n v_n = 0$

Also since $v_1, v_2, \cdots v_n$ are L.I. on I, we have $c_2 = c_3 = \cdots = c_n = 0$

and ③ $\Rightarrow c_1 = 0$

Thus the function in ② from a basis for the soln of

$L(y) = 0$ on I

Theorem: 10

If φ_1 is a soln of $L(y) = y'' + a_1(x) y' + a_2(x) y = 0$ on an $\rightarrow$ ①

interval I and $\varphi_1(x) \neq 0$ on I a second soln φ_2 of ① on I

is given by $\varphi_2(x) = \varphi_1(x) \int_{x_0}^{x} \frac{1}{[\varphi_1(s)]^2} \exp\left[-\int_{x_0}^{x} a_1(t)\,dt\right] ds$ The function

φ_1, φ_2 form a basis for the soln of ① on I

Proof:

Given $L(y) = y'' + a_1(x) y' + a_2(x) y = 0$

If φ_1 is a soln on I.

We have,

$L[u\varphi_1] = (u\varphi_1)'' + a_1 (u\varphi_1)' + a_2 (u\varphi_1)$

$= u''\varphi_1 + 2u'\varphi_1' + u\varphi_1'' + a_1 u'\varphi_1 + a_1 u\varphi_1' + a_2 u\varphi_1$

$= u''\varphi_1 + 2(\varphi_1' + a_1\varphi_1) u' + [\varphi_1'' + a_1\varphi_1' + a_2\varphi]$

$= u''\varphi_1 + (2\varphi_1' + a_1\varphi_1) u' + u(0)$ (since $L(\varphi_1) = 0$)

$L(u\varphi_1) = u''\varphi_1 + (2\varphi_1' + a_1\varphi_1) u'$

Let $v = u'$ and u is such that $L(u\varphi_1) = 0$

$\Rightarrow \varphi_1 v' + 2(\varphi_1' + a_1\varphi_1) v = 0 \rightarrow$ ②

This is linear eqn of order one and can always be solved explicity.

provided $\varphi_1(x) \neq 0$ on I

② $\times \varphi_1$ we get,

$\varphi_1^2 v' + (2\varphi_1\varphi_1' + a_1\varphi_1^2) v = 0 \rightarrow$ ③

$$\Rightarrow \varphi_1^2 v' + 2\varphi_1\varphi_1' v + a_1\varphi_1^2 v = 0$$

$$(\varphi_1^2 v)' + a_1(\varphi_1^2 v) = 0$$

$$\varphi_1^2(x)\, v(x) = c\exp\left[-\int_{x_0}^{x} a_1(t)\,dt\right]$$

$\frac{dy}{dx} + Py = Q$ (61)

$P = a_1$ $\quad Q = 0$

$e^{\int P dt}$ $\int a_1 dt$

$= e$

$y e^{\int a_1 dt} = \int 0 + c$ $\quad \phi_1^2 w = ce^{-\int a_1 dt}$

Where x_0 is a point in I and c is a constant.

since any constant multiple of a soln of (3) is again a soln, we see that.

$$v(x) = \frac{1}{[\varphi_1(x)]^2}\exp\left[-\int_{x_0}^{x} a_1(t)\,dt\right]$$ is a soln of (3) and also (2)

$\therefore$ The two independent soln of $L(y) = y'' + a_1(x)y' + a_2(x)y = 0$ (4)

on I are φ_1 and φ_2 $\quad \phi_2 = u\phi_1$ $\quad \cancel{v = u\phi_1 = \phi_2}$ $\quad \& v = u'$ $\Rightarrow u = \int v$

where,
$$\varphi_2(x) = \varphi_1(x)\int_{x_0}^{x} \frac{1}{[\varphi_1(s)]^2}\exp\left[-\int_{x_0}^{x} a(t)\,dt\right]ds$$

Example:

Consider the eqn $y'' - \left(\frac{2}{x}\right)y = 0$, $0 < x < \infty$. Its easy to verify that the φ_1 given by $\varphi_1(x) = x^2$ is a soln $0 < x < \infty$. $\quad 2/x^2$

Since this function not vanish on this interval, There is another independent soln φ_2 of the form $\varphi_2 = u\varphi_1$

If $v = u'$ then satisfies.

$$x^2 v' + 4xv = 0 \quad (\text{or}) \quad xv' + 4v = 0$$

A soln for this is given by

$$v(x) = x^{-4} \qquad 0 < x < \infty$$

$x\frac{dv}{dx} = -4v$

$\frac{dv}{v} = -4\frac{dx}{x}$

$\log v + 4\log x = \log c$

$v x^4 = c$

$\boxed{v = x^{-4}}$

$u = \frac{x^{-4+1}}{-4+1} = \frac{x^{-3}}{-3} = -\frac{1}{3x^3}$

and therefore a choice for u is $u(x) = -\frac{1}{3x^3}$, $0 < x < \infty$.

This leads to $\varphi_2(x) = \frac{-1}{3x}$, $0 < x < \infty$

but since any constant times a soln is a soln we may as well choose for a second soln $\varphi_2(x) = x^{-1}$. Thus x^2, x^{-1} form a basis for the solns on $0 < x < \infty$.

Method of finding one soln of a Second order homogeneous eqn (6)

Put the eqn in the form.

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\,y = 0$$

Rule: 1

$y = e^{mx}$ is a soln if $m^2 + m\alpha_1 + \alpha_2 = 0$.

In particular,

$y = x$ is a soln if $a_1 + a_2 x = 0$

$y = x^2$ is a soln if $2 + a_1 x + a_2 x^2 = 0$

$y = 1/x$ is a soln if $2 - a_1 x + a_2 x^2 = 0$

As an illustration the eqn

$x\frac{d^2y}{dx^2} - (2x-1)\frac{dy}{dx} + (x-1)y = 0$ can be put in the form

$$\frac{d^2y}{dx^2} - \left(\frac{2x-1}{x}\right)\frac{dy}{dx} + \left(\frac{x-1}{x}\right)y = 0$$

with $a_1 = -\left(\frac{2x-1}{x}\right)$, $a_2 = \left(\frac{x-1}{x}\right)$ and $1 + a_1 + a_2 = 0$

$\therefore$ $y = e^{x}$ is a soln.

The second soln can be found using reduction of order m

3.a) consider the eqn $x^2y'' - 7xy' + 15y = 0$ Such That $\phi_1(x) = x^3$ $(x>0)$ is a soln of the eqn and find a second independent soln

Soln:

Given eqn is $x^2y'' - 7xy' + 15y = 0 \rightarrow (1)$

$\phi_1(x) = x^3$

$\Rightarrow \phi_1'(x) = 3x^2, \quad \phi_1''(x) = 6x$

$$x^2\phi_1'' - 7x\phi_1' + 15\phi = x^2(6x) - 7x(3x^2) + 15x^3$$

$$= 6x^3 - 7x^3 + 15x^3$$

$$x^2\phi_1'' - 7x\phi_1' + 15\phi = 0$$

$\therefore$ ϕ satisfies the eqn.

Let the second soln φ_2 be of the form.

$$\varphi_2 = u\varphi_1 = ux^3$$

$$\varphi_2' = u.3x^2 + x^3.u'$$

$$\varphi_2'' = u.6x + 3x^2u' + 3x^2u' + x^3u''$$

$$\varphi_2'' = x^3u'' + 6x^2u' + 6xu$$

Sub in ①

$$x^2(x^3u'' + 6x^2u' + 6xu) - 7x(x^3u' + 3x^2u) + 15x^3u = 0$$

$$x^5u'' + 6x^4u' + 6x^3u - 7x^4u' - 21x^3u + 15x^3u = 0$$

$$x^5u'' - x^4u' = 0$$

$$\Rightarrow \quad u'' - \frac{1}{x}u' = 0$$

$$v' - \frac{1}{x}v = 0 \qquad \text{where } v = u'$$

$\therefore \quad v = x, \quad u' = x, \quad u = x^2/2$

we can take, $u = x^2$

$$\varphi_2 = \varphi_1.u = x^3.x^2 = x^5$$

also $W(\varphi_1, \varphi_2) = 2x^7 \neq 0 \quad (\because x > 0)$

$\therefore$ The two independent solns are $\varphi_1 = x^3$, $\varphi_2 = x^5$.

3.b) one soln of $x^3y''' - 3x^2y'' + 6xy' - 6y = 0$ for $x > 0$ is $\varphi_1(x) = x$

find a basis for the soln for $x > 0$.

Soln:

$\varphi_1(x) = x$ is a first soln.

$\varphi_2(x) = u\varphi_1 = x.u$ be a second soln of the eqn

$$\varphi_2'(x) = xu' + u$$

$$\varphi_2''(x) = xu'' + 2u'$$

$$\varphi_2'''(x) = xu''' + 3u''$$

Sub in Given eqn.

$$x^3(xu''' + 3u'') - 3x^2(xu'' + 2u') + 6x(xu' + u) - 6xu = 0$$

i.e) $x^4u''' = 0$

$$u''' = 0.$$

The characteristic Polynomial is

$$r^3 = 0$$

The roots are 0,0,0

$\therefore$ The three independent solns are $1, x, x^2$

The three solns of the given eqn are x, x^2, x^3

Also, $W(x, x^2, x^3) = 2x^3 \neq 0 \quad (x>0)$

Hence these solns are linearly independent.

Pg, 121
Sn
1 b, c
d, e, f
Hw.
(d) 14
12

Section-6

The non-homogeneous eqn.

Theorem: 11 (Variation of constants method extended to eqn with variance coefficients)

Let b be continuous on an interval I and let $\varphi_1, \varphi_2 \ldots \varphi_n$ be a basis for the solns of $L(y)=0$ on I. Every soln ψ of $L(y) = b(x)$ can be written as $\psi = \psi_p + c_1\varphi_1 + \cdots + c_n\varphi_n$ where ψ_p is a Particular soln of $L(y) = b(x)$ and $c_1, c_2 \ldots c_n$ are constants. Every soln ψ is a soln of $L(y) = b(x)$.

A Particular soln ψ_p is given by

$$\psi_p(x) = \sum_{k=1}^{n} \varphi_k(x) \int_{x_0}^{x} \frac{W_k(t)\, b(t)}{W(\varphi_1, \varphi_2 \ldots \varphi_n)(t)}\, dt$$

Proof:

Consider the eqn $L(y) = y^{(n)} + a_1(x) y^{(n-1)} + \cdots + a_n(x) y = b(x) \rightarrow (1)$

Where $a_1, a_2 \ldots a_n, b$ are continuous function on I.

W.K.T when all a_k's are constants. The eqn can be solved by the method of variation of constants.

The method does not depend on the fact that the a_k are constants and is therefore valid for the eqn (1)

If ψ_p is a particular soln of (1) any soln ψ has the form,

$$\psi = \psi_p + c_1\varphi_1 + c_2\varphi_2 + \cdots + c_n\varphi_n.$$

where $c_1, c_2 \dots c_n$ are constants and $\varphi_1, \varphi_2 \dots \varphi_n$ is a basis for the soln of $L(y)=0$. (65)

Every ψ is a soln of $L(y) = b(x)$.

A particular soln ψ_p can be found having the form

$$\psi_p = u_1\varphi_1 + u_2\varphi_2 + \dots + u_n\varphi_n$$

where $u_1, u_2 \dots u_n$ are function satisfying,

$$u_1'\varphi_1 + u_2'\varphi_2 + \dots + u_n'\varphi_n = 0$$

$$u_1'\varphi_1' + u_2'\varphi_2' + \dots + u_n'\varphi_n' = 0$$

$$\vdots$$

$$u_1'\varphi_1^{(n-2)} + u_2'\varphi_2^{(n-2)} + \dots + u_n'\varphi_n^{(n-2)} = 0$$

$$u_1'\varphi_1^{(n-1)} + u_2'\varphi_2^{(n-1)} + \dots + u_n'\varphi_n^{(n-1)} = b$$

If x_0 is any point on I, take,

$$u_k(x) = \int_{x_0}^{x} \frac{w_k(t)\, b(t)}{w(\varphi_1, \varphi_2 \dots \varphi_n)(t)}\, dt \qquad (k = 1, 2 \dots n)$$

and then ψ_p has the form,

$$\psi_p(x) = \sum_{k=1}^{n} \varphi_k(x) \int_{x_0}^{x} \frac{w_k(t)\, b(t)}{w(\varphi_1, \varphi_2 \dots \varphi_n)(t)}\, dt$$

Here $w(\varphi_1, \varphi_2 \dots \varphi_n)$ is the wronskian of the basis $\varphi_1, \varphi_2 \dots \varphi_n$ and w_k is the determinant obtained from $w(\varphi_1, \varphi_2 \dots \varphi_n)$ by replacing the k^{th} column $(\varphi_k, \varphi_k' \dots \varphi_k^{(n-1)})$ by $(0, 0 \dots 0, 1)$

Problems.

1.a) Find all soln of the eqn $y'' - \frac{2}{x^2} y = x$, $0 < x < \infty$ and given $\varphi_1(x) = x^2$ is one soln of $y'' - \frac{2}{x^2} y = 0$

Soln:

$$y'' - \frac{2}{x^2} y = x \rightarrow (1)$$

and

$$y'' - \frac{2}{x^2} y = 0 \rightarrow (2)$$

Now, $\varphi_1(x) = x^2$, $\varphi_2(x) = x^{-1}$

A soln ψ_p of the non homogeneous eqn has the form.

$$\psi_p = u_1\varphi_1 + u_2\varphi_2$$

$$\psi_p = u_1 x^2 + u_2 x^{-1} \rightarrow (3)$$

Where u_1', u_2' satisfy.

$$x^2 u_1' + x^{-1} u_2' = 0$$

$$2x u_1' - x^{-2} u_2' = x$$

Now,

$$W(\varphi_1,\varphi_2) = \begin{vmatrix} x^2 & x^{-1} \\ 2x & -x^{-2} \end{vmatrix}$$

$$= x^2(-x^{-2}) - 2x \cdot x^{-1}$$

$$= -1-2$$

$$W(\varphi_1,\varphi_2) = -3$$

$$\therefore u_1' = \frac{\begin{vmatrix} 0 & 1/x \\ x & -1/x^2 \end{vmatrix}}{-3} = \frac{0 - x \cdot \frac{1}{x}}{3}$$

$$u_1' = \frac{1}{3} \rightarrow (4)$$

$$u_2' = \frac{\begin{vmatrix} x^2 & 0 \\ 2x & x \end{vmatrix}}{-3} = \frac{x^3}{-3}$$

$$u_2' = -\frac{x^3}{3} \rightarrow (5)$$

for u_1, u_2 we may take,

$(4) \Rightarrow \quad u_1 = \frac{x}{3}$

$(5) \Rightarrow \quad u_2 = -\frac{x^4}{12}$

$(3) \Rightarrow \quad \psi_p = \left(\frac{x}{3}\right)x^2 + \left(-\frac{x^4}{12}\right)x^{-1}$

$$\psi_p = \frac{x^3}{3} - \frac{x^3}{12} = \frac{x^3}{4}$$

Every soln φ of (1) then has form.

$$\varphi(x) = \frac{x^3}{4} + c_1x^2 + c_2x^{-1}$$

b) one soln of $x^2y'' - xy' + y = 0$ $(x>0)$ is $\varphi_1(x) = x$.

find the soln ψ of $x^2y'' - xy' + y = x^2$ satisfying $\psi(1) = 1$, $\psi'(1) = 0$

Soln:

Given $\varphi_1(x) = x$,

Let $\varphi_2 = u\varphi_1 = ux$ be a second soln of the homogeneous eqn

$$x^2y'' - xy' + y = 0$$

$$x^2(xu)'' - x(xu)' + xu = 0$$

$$x^2[xu'' + 2u'] - x[xu' + u] + xu = 0$$

$$x^3u'' + 2x^2u' - x^2u' = 0$$

$$xu'' + u' = 0$$

$$xv' + v = 0 \quad \text{by Taking } v = u'$$

Solving woget, $v = \frac{1}{x}$

$$\therefore u = \int v\,dx = \log x$$

$$\varphi_2(x) = x\log x$$

A soln ψ_p of the non homogeneous eqn is

$$\psi_p = u_1 x + u_2 x\log x$$

where u_1' and u_2' satisfy

$$u_1'x + u_2' x\log x = 0$$

$$u_1' + u_2'[1 + \log x] = x^2$$

$$\therefore W(\varphi_1, \varphi_2) = \begin{vmatrix} x & x\log x \\ 1 & 1+\log x \end{vmatrix} = x(1+\log x) - x\log x$$

$$= x + x\log x - x\log x$$

$$W(\varphi_1, \varphi_2) = x$$

$$u_1' = \frac{\begin{vmatrix} 0 & x\log x \\ x^2 & 1+\log x \end{vmatrix}}{x} = -x^2\log x \Rightarrow u_1 = -\int x^2\log x\,dx$$

$$u_1 = -\frac{x^3}{3}\log x + \frac{x^3}{9}$$

$$u_2' = \frac{\begin{vmatrix} x & 0 \\ 1 & x^2 \end{vmatrix}}{x} = \frac{x^3}{x} = x^2 \Rightarrow u_2 = \frac{x^3}{3}$$

$\therefore \psi = \psi_p + c_1\varphi_1 + c_2\varphi_2$

$= x\left[-\frac{x^3}{3}\log x + \frac{x^3}{9}\right] + (x\log x)\frac{x^3}{3} + c_1 x + c_2 x\log x$

$\psi = c_1 x + c_2 x\log x + \frac{x^4}{9}$

Given $\psi(1) = 1$

$c_1 + \frac{1}{9} = 1$

$c_1 = \frac{8}{9}$

and $\psi'(1) = 0$

$c_1 + c_2 + \frac{4}{9} = 0$

$\frac{8}{9} + c_2 + \frac{4}{9} = 0$

$c_2 + \frac{4}{3} = 0$

$c_2 = -\frac{4}{3}$

$\therefore \psi = \frac{8}{9}x - \frac{4}{3}x\log x + \frac{x^4}{9}$

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1.c) Consider the eqn $y'' + a_1(x)y' + a_2(x)y = 0$, where a_1, a_2 are contin on some interval I containing x_0. Suppose φ_1 is a soln suc $\varphi_1(x) \neq 0$ for all x in I.

(a) S.T there is a second soln φ_2 on I such that, $W(\varphi_1,\varphi_2)(x_0) = 1$

(b) Compute such a φ_2 interms of φ_1 by solving the first order
$\varphi_1(x)\varphi_2'(x) - \varphi_1'(x)\varphi_2(x) = \exp\left[-\int_{x_0}^{x} a_1(t)\,dt\right]$

Soln:

(a) for some $x_0 \in I$

let $\varphi_2(x_0) = 0$ and $\varphi_2'(x_0) = \frac{1}{\varphi_1(x_0)}$

Then

$$W(\varphi_1,\varphi_2)(x_0) = \begin{vmatrix} \varphi_1(x_0) & \varphi_2(x_0) \\ \varphi_1'(x_0) & \varphi_2'(x_0) \end{vmatrix}$$

$$= \begin{vmatrix} \varphi_1(x_0) & 0 \\ \varphi_1'(x_0) & \frac{1}{\varphi_1(x_0)} \end{vmatrix} = \varphi_1(x_0)\cdot\frac{1}{\varphi_1(x_0)}$$

$W(\varphi_1,\varphi_2)(x_0) = 1$

$\therefore \varphi = \varphi_p + c_1\varphi_1 + c_2\varphi_2$

$= x\left[-\frac{x^3}{3}\log x + \frac{x^3}{9}\right] + (x\log x)\frac{x^3}{3} + c_1 x + c_2 x\log x$

$\varphi = c_1 x + c_2 x\log x + \frac{x^4}{9}$

Given $\varphi(1) = 1$

$c_1 + \frac{1}{9} = 1$

$c_1 = \frac{8}{9}$

and $\varphi'(1) = 0$

$c_1 + c_2 + \frac{4}{9} = 0$

$\frac{8}{9} + c_2 + \frac{4}{9} = 0$

$c_2 + \frac{4}{3} = 0$

$c_2 = -\frac{4}{3}$

$\therefore \varphi = \frac{8}{9}x - \frac{4}{3}x\log x + \frac{x^4}{9}$

pg 122 ex 5

1.c) consider the eqn $y'' + a_1(x)y' + a_2(x)y = 0$, where a_1, a_2 are continuous on some interval I containing x_0. Suppose φ_1 is a soln such that $\varphi_1(x) \neq 0$ for all x in I.

(a) S.T there is a second soln φ_2 on I such that, $W(\varphi_1, \varphi_2)(x_0) = 1$.

(b) compute such a φ_2 interms of φ_1 by solving the first order eqn

$\varphi_1(x)\varphi_2'(x) - \varphi_1'(x)\varphi_2(x) = \exp\left[-\int_{x_0}^{x} a_1(t)\,dt\right]$

Soln:

(a) for some $x_0 \in I$

let $\varphi_2(x_0) = 0$ and $\varphi_2'(x_0) = \frac{1}{\varphi_1(x_0)}$

Then
$$W(\varphi_1,\varphi_2)(x_0) = \begin{vmatrix} \varphi_1(x_0) & \varphi_2(x_0) \\ \varphi_1'(x_0) & \varphi_2'(x_0) \end{vmatrix}$$

$$= \begin{vmatrix} \varphi_1(x_0) & 0 \\ \varphi_1'(x_0) & \frac{1}{\varphi_1(x_0)} \end{vmatrix} = \varphi_1(x_0)\cdot\frac{1}{\varphi_1(x_0)}$$

$W(\varphi_1, \varphi_2)(x_0) = 1$

b) Given $\varphi_1(x)\varphi_2'(x) - \varphi_1'(x)\varphi_2(x) = \exp[-\int_{x_0}^{x} a_1(t)dt]$

This is linear in φ_2

$\therefore \varphi_2'(x) - \frac{\varphi_1'(x)}{\varphi(x)} \cdot \varphi_2(x) = \frac{1}{\varphi(x)} A(t)$, where $A(t) = \exp[-\int_{x_0}^{x} a_1(t)dt]$

$\therefore \varphi_2(x) = \varphi_1(x) \int_{x_0}^{x} \frac{1}{|\varphi(x)|^2} A(t)dt$

1.d) Two solns of $x^3y''' - 3xy' + 3y = 0$ $(x>0)$ are $\varphi_1(x) = x$, $\varphi_2(x) = x^3$ use this information to find a Third independent soln.

Soln:

Method of Successive Reduction of orders

Step: 1 First stage reduction

$$x^3y''' - 3xy' + 3y = 0 \rightarrow (1)$$

$\varphi_1(x) = x$ is a soln of (1)

Let $\varphi_2(x) = u\varphi_1 = ux$ be The Second soln.

Sub in (1)

$$x^3(ux)''' - 3x(ux)' + 3(ux) = 0$$

$$x^3(u'''x + 3u'') - 3x(u'x + u) + 3ux = 0$$

$$x^4u''' + x^3 3u'' - x^2 3u' = 0$$

$$x^2(x^2u''' + 3xu'' - 3u') = 0$$

$$x^2u''' + 3xu'' - 3u' = 0$$

Taking $v = u'$

$$x^2v'' + 3xv' - 3v = 0$$

This is a second order differential eqn, we again reduce its order by 1

Step: 2 Second stage reduction

$$x^2v'' + 3xv' - 3v = 0 \rightarrow (2)$$

We have $\varphi_2 = u\varphi_1$

$u = \frac{\varphi_2}{\varphi_1} = \frac{x^3}{x} = x^2$ $(\because \varphi_1(x) = x, \quad \varphi_2(x) = x^3)$

~~Bxx~~ ~~xxx~~

But $v = u' = 2x$

$v_2 = 2xw$

Sub in (2) we get,

$$x^2(2xw)'' + 2x(2xw)' - 3(2xw) = 0$$

$$2x^2(xw'' + 2w') + 6x(xw' + w) - 6xw = 0$$

$$2x^3 w'' + 10x^2 w' = 0$$

$$xw'' + 5w' = 0$$

Taking $z = w'$ we get,

$$xz' + 5z = 0$$

Solving this we get,

$$z = \frac{1}{x^5}$$

Now get the soln of the original eqn by back substitution in the following order. $z \to w \to v \to u \to \phi$

$z = w'$, $\therefore w = \int z\,dx = \int \frac{dx}{x^5} = \frac{-4}{x^4}$

$v_2 = 2xw = \frac{-8}{x^3}$

$v = u'$, $u = \int v\,dx = \int \frac{-8}{x^3}dx = \frac{24}{x^2}$

$\phi_2 = ux = \frac{24}{x}$

Omiting constant we get $\phi(x) = \frac{1}{x}$ is the third soln of the given original eqn.

We can show that $w(\frac{1}{x}, x, x^2) = 8 \neq 0$

$\therefore$ The solns $\frac{1}{x}, x, x^3$ are linearly independent.

1.e)

(a) S.T there is a basis ϕ_1, ϕ_2 for the solns of $x^2y'' + 4xy' + (2+x^2)y = 0$ $(x>0)$ of the form $\phi_1(x) = \frac{\psi_1(x)}{x^2}$, $\phi_2(x) = \frac{\psi_2(x)}{x^2}$

6. Find all the solns of $x^2y''+4xy'+(2+x^2)y=x^2$ for $x>0$.

Soln:

(a) If φ is a soln of $x^2y''+4xy'+(2+x^2)y=0 \rightarrow$ (1)

let $\varphi=\frac{v}{x^2}$ be another soln. v being some function of x.

$$\varphi'=\frac{x^2v'-2vx}{x^4} \qquad \frac{xv'-2v}{x^3}$$

$$\varphi''=\frac{x^2v''-4xv'+6v}{x^4}$$

Sub in (1)

$$x^2\left[\frac{x^2v''-4xv'+6v}{x^4}\right]+4x\left[\frac{xv'-2v}{x^3}\right]+(2+x^2)\frac{v}{x^2}=0$$

ie) $v''+v=0$

The characteristic polynomial

$$r^2+1=0$$
$$r=\pm i$$

$\therefore$ The two independent solns are $\cos x$, $\sin x$.

Hence $\varphi_1=\frac{\cos x}{x^2}$, $\varphi_2=\frac{\sin x}{x^2}$

$\therefore$ Here $\psi_1(x)=\cos x$; $\psi_2(x)=\sin x$

Any particular soln ψ_p of the eqn

$x^2y''+4xy'+(x^2+2)y=x^2$ is of the form

$$\psi_p=u_1\varphi_1+u_2\varphi_2$$

$$\psi_p=u_1\frac{\cos x}{x^2}+u_2\frac{\sin x}{x^2}$$

where u_1' and u_2' satisfy

$$u_1'\frac{\cos x}{x^2}+u_2'\frac{\sin x}{x^2}=0$$

$$u_1'\left\{\frac{-x\sin x-2\cos x}{x^3}\right\}+u_2'\left\{\frac{x\cos x-2\sin x}{x^3}\right\}=x^2$$

$$W(\varphi_1,\varphi_2)=\begin{vmatrix}\varphi_1 & \varphi_2\\ \varphi_1' & \varphi_2'\end{vmatrix}$$

$$W(\varphi_1,\varphi_2)=\frac{1}{x^4}$$

$$u_1' = \frac{\begin{vmatrix} 0 & \frac{\sin x}{x^2} \\ x & \frac{x\cos x - 2\sin x}{x^3} \end{vmatrix}}{\frac{1}{x^4}} = -x^4\sin x$$

$$u_2' = \frac{\begin{vmatrix} \frac{\cos x}{x^2} & 0 \\ \frac{-x\sin x - 2\cos x}{x^3} & x^2 \end{vmatrix}}{\frac{1}{x^4}} = x^4\cos x$$

$u_1 = -\int x^4 \sin x\,dx$, $u_2 = \int x^4 \cos x\,dx$

using bernoulli's formula for integration.

$u_1 = -[x^4(-\cos x) - 4x^3(-\sin x) + 12x^2(\cos x) - 24x\sin x - 24\cos x]$

$u_1 = x^4\cos x - 4x^3\sin x - 12x^2\cos x + 24x\sin x + 24\cos x$

$u_2 = x^4\sin x - 4x^3(-\cos x) + 12x^2(-\sin x) - 24x\cos x + 24\sin x$

$u_2 = x^4\sin x + 4x^3\cos x - 12x^2\sin x - 24x\cos x + 24\sin x$

$\therefore \psi_p = u_1\phi_1 + u_2\phi_2$

$= (x^4\cos x - 4x^3\sin x - 12x^2\cos x + 24x\sin x + 24\cos x)\frac{\cos x}{x^2}$
$+ [x^4\sin x + 4x^3\cos x - 12x^2\sin x - 24x\cos x + 24\sin x]\frac{\sin x}{x^2}$

$\psi_p = x^2 + \frac{24}{x^2} - 12$

$\psi = c_1\frac{\cos x}{x^2} + c_2\frac{\sin x}{x^2} + x^2 + \frac{24}{x^2} - 12$

1.f) Reduction of order method to find a Particular soln.

(a) consider the eqn $y'' + a_1(x)y' + a_2(x)y = b(x)$. where a_1, a_2, b are continuous on some interval I. suppose ϕ_1 is a soln of $L(y) = 0$ such that $\phi_1(x) \neq 0$ $\forall$ x in I. S.T. There is a particular soln ψ_p of $L(y) = b(x)$ of the form $\psi_p = u_p\phi_1$ where $v_p = u_p'$ is a particular soln of the first order eqn

$\phi_1(x)v' + [2\phi_1'(x) + a_1(x)\phi_1(x)]v = b(x)$.

5. using The idea in (a) to find all solns of $x^2y'' - xy' + y = x^2$ for $x>0$.

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Soln:

a) let $\psi_p = u_p\varphi_1$ be any particular soln of $L(y) = b(x)$ given, where φ_1 is a soln of $L(y) = 0$.

$$\psi_p' = u_p\varphi_1' + u_p'\varphi_1$$

$$\psi_p'' = u_p\varphi_1'' + 2u_p'\varphi_1' + u_p''\varphi_1$$

Sub in The eqn $L(y) = b(x)$ we obtain,

Since $L(\varphi_1) = 0$

$$u_p(\varphi_1'' + a_1\varphi_1' + a_2\varphi_1) + u_p''\varphi_1 + (2\varphi_1' + a_1\varphi_1)u_p' = b(x)$$

where $u_p' = v_p$

ie) $u_p''\varphi_1 + (2\varphi_1' + a_1\varphi_1)u_p' = b(x)$

$$v_p'\varphi_1(x) + 2(\varphi_1'(x) + a_1(x)\varphi_1)v_p = b(x)$$

(b) Consider The eqn

$$x^2y'' - xy' + y = x^2$$

one soln of The homogeneous eqn $x^2y'' - xy' + y = 0$ is $\varphi_1(x) = x$.

Now,
$$\psi_p = u_p\varphi_1 = u_p x$$

$$\psi_p' = u_p'x + u_p$$

$$\psi_p'' = u_p''x + 2u_p'$$

Sub in The eqn $x^2y'' - xy' + y = x^2$

we get,

$$x^2[xu_p'' + 2u_p'] - x[xu_p' + u_p] + xu_p = x^2$$

$$x^3u_p'' + x^2u_p' = x^2$$

$$xu_p'' + u_p' = 1$$

Taking $u_p' = v_p$

$$xv_p' + v_p = 1$$

ie) $\frac{d}{dx}(xv_p) = 1$

$$xv_p = 1$$

$$\therefore u_p = \int v_p\,dx = \int 1.dx = x$$

and $\psi_p = u_p\varphi_1 = x^2$

Now we find the two indepe...

$x^2y''-xy'+y=0$

Since $\varphi_1(x)=x$ is a soln

we take $\varphi_2=u\varphi_1=ux$

Sub we get,

$$x^2(ux)''-x(ux)'+ux=0$$

$$x^2(u''x+2u')-x(u'x+u)+ux=0$$

$$x^3u''+x^2u'=0$$

$$xu''+u'=0$$

Taking $u'=v$

We get, $xv'+v=0$

$$\frac{d}{dx}(xv)=0$$

$$xv=1$$

$$v=\frac{1}{x}$$

$$u=\int v\,dx=\int\frac{1}{x}dx$$

$$u=\log x$$

$$\therefore \varphi_2=ux=x\log x$$

$\therefore$ The two linearly independent solns of the homogeneous eqn

$x^2y''-xy'+y=0$ are $\varphi_1(x)=x$, $\varphi_2(x)=x\log x$

We have,

$$\psi_p=xu_p=x^2$$

$\therefore$ The most general soln is

$$\psi=\psi_p+c_1\varphi_1+c_2\varphi_2$$

$$\psi=x^2+c_1x+c_2x\log x.$$

1.g) Consider the eqn $y''+y=b(x)$, where b is a continuous fun on $1\le x<\infty$ satisfying $\int_1^\infty |b(t)|\,dt<\infty$

(a) S.T a particular soln ψ_p is given by $\psi_p(x)=\int_1^x \sin(x-t)\,b(t)\,dt$

(b) S.T any soln is bounded on $1\le x<\infty$

Soln:

The characteristic polynomial of $y''+y=0$ is $r^2+1=0$

(i) $r^2=-1 \Rightarrow r=\pm i$

$\therefore$ we take $\varphi_1(x)=\cos x$, $\varphi_2(x)=\sin x$ as two linearly independent solns.

Any particular soln ψ_p of $y''+y=b(x)$ has the form

$\psi_p = u_1\varphi_1 + u_2\varphi_2$

Where u_1' and u_2' satisfy

$$u_1'\varphi_1 + u_2'\varphi_2 = 0$$

$$u_1'\varphi_1' + u_2'\varphi_2' = b(x)$$

$$\therefore u_1'\cos x + u_2'\sin x = 0$$

$$-u_1'\sin x + u_2'\cos x = b(x)$$

$$u_1' = \frac{\begin{vmatrix} 0 & \sin x \\ b(x) & \cos x \end{vmatrix}}{W(\varphi_1,\varphi_2)} = \frac{-b(x)\sin x}{1}$$

$$u_1' = -b(x)\sin x$$

$$u_2' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & b(x) \end{vmatrix}}{W(\varphi_1,\varphi_2)} = \frac{b(x)\sin x}{1}$$

$$u_2' = b(x)\sin x.$$

$$\psi_p = \varphi_1 u_1 + \varphi_2 u_2 = \cos x\left[\int_1^x -b(t)\sin t\,dt\right] + \sin x\int_1^x b(t)\cos t\,dt$$

$$= \int_1^x (\sin x\cos t - \cos x\sin t)\,b(t)\,dt$$

$$= \int_1^x \sin(x-t)\,b(t)\,dt$$

$$\therefore \psi = c_1\cos x + c_2\sin x + \int_1^\infty \sin(x-t)\,b(t)\,dt$$

(b) $|\psi| \le (c_1+c_2) + \int_1^\infty |b(t)|\,dt$

$< \infty$

$\therefore$ ψ is bounded on $1 \le x < \infty$